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# The Moving-Estimates Test for Parameter Stability-Revised



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## Abstract

In this paper a new class of tests for parameter stability, the moving-estimates (ME) test, is proposed. It is shown that the asymptotic null distribution of the ME test is determined by the boundary-crossing probabilities of the increments of a Brownian bridge. It is also shown that under a broad class of alternatives the ME test is consistent and has non-trivial local power in general. Our simulations also show that the proposed test has power superior to other competing tests when parameters are temporarily instable.

**JEL Classification Number:** 211

**Keywords:** Boundary-crossing probability; Brownian bridge; Fluctuation test; Functional central limit theorem, Moving estimate, Moving-estimates test; Structural change; Wiener process.

## 1 Introduction

Testing for parameter constancy has recently been receiving much attention in the econometric literature. Contributions to this topic include the fluctuation (FL) test of Sen [18] and Ploberger, Krämer, & Kontrus [17], the maximal-Wald test of Hawkins [13], the maximal-likelihood-ratio (LR) and Lagrange-Multiplier (LM) tests of Andrews [1], and the exponentially weighted Wald, LM, and LR tests of Andrews & Ploberger [2] and Andrews, Lee, & Ploberger [3]. In contrast with traditional testing procedures such as the Chow [7] test, a novel feature of these new tests is that no prior knowledge of the location of change point is required. Under quite general conditions, these new test statistics have well defined asymptotic distributions in the space of continuous functions, and their critical values can be determined by the well-known boundary-crossing probability formulae of associated limiting processes or by simulations. These tests have also been extended to models with trending regressors, e.g., Chu & White [9] and Hansen [12].

In this paper we introduce a *new* class of tests, the moving-estimates (ME) test, for parameter stability. In contrast with the maximal or exponentially weighted Wald, LM, and LR tests which are formulated against a specific alternative, typically a one-time structural change, the ME test is a "general" test constructed without assuming any specific alternative in mind. Comparing to the FL test which is based on *recursive* estimates calculated from a sequence of subsamples of increasing size, the ME test is determined by the fluctuation of *moving* estimates computed from a sequence of subsamples of the same size. Note also that the ME test is different from the "homogeneity test" of Brown, Durbin, & Evans [6] and the moving (rolling) $t$-tests of Banerjee, Lumsdaine, & Stock [4]. The ME test is particularly sensitive to the alternative of double structural changes where parameters temporarily deviate from a "normal" level because it can be interpreted as the maximal-LR test under this alternative. Moreover, as moving estimates implement a locally weighted regression, the ME test is a nonparameteric test for a general nonconstant mean function.

In this paper it is shown that fluctuations of moving estimates (in terms of their deviations from the full-sample estimate) converge weakly to the increments of a Brownian bridge under the null hypothesis of constant parameters. Hence, the asymptotic null distribution of the ME test is determined by the boundary-crossing probability of this limiting process. The limiting process under the null is non-standard, and analytic expressions of its boundary-crossing probabilities, corresponding to the cases where moving subsamples are at least half of the entire sample, have recently been derived in Chu, Hornik, & Kuan [8] and Hornik [14], from which asymptotic critical values can be easily calculated. For other bandwidths of moving subsamples, asymptotic critical values can be obtained from simulations. It is also shown that under a broad class of alternatives, the ME test is consistent and has non-trivial local power in general. Our simulations also indicate that the ME tests have power superior to other competing tests when parameters are temporarily instable.

This paper is organized as follows. We introduce the ME test for the location model in section 2. The asymptotic null distributions are derived in section 3. Extension to multiple regression is discussed in section 4. We report simulation results in section 5. Section 6 summarizes the paper. All proofs are deferred to the Appendix.

## 2 The Moving-Estimates Test

Consider the location model

$$y_i = \mu_i + \epsilon_i, \quad i = 1, 2, \ldots, T.$$

We assume that $\{\epsilon_i\}$ is a sequence of random variables obeying a functional central limit theorem (FCLT):

$$\left( \frac{1}{\sigma\sqrt{T}} \sum_{i=1}^{[Tt]} \epsilon_i, 0 \leq t \leq 1 \right) \Rightarrow W \tag{1}$$

as $T \to \infty$, where

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\Big[\Big(\sum_{i=1}^{T} \epsilon_i\Big)^2\Big],$$

$[Tt]$ is the integer part of $Tt$, $\Rightarrow$ denotes weak convergence (of the associated probability measures), and $W$ is the standard Wiener process. For more details about weak convergence and FCLT we refer to Billingsley [5]. We note that (1) holds under fairly general conditions and that (1) remains valid when $\sigma$ is replaced by a consistent estimator $\hat{\sigma}$; see e.g., Wooldridge & White [19].

Under the null hypothesis that the parameter is a constant, $\mu_i = \mu_0$ for all $i$. The sequence of recursive estimates of $\mu_0$ is

$$\hat{\mu}_k = \frac{1}{k} \sum_{i=1}^{k} y_i, \quad k = 1, \cdots, T. \tag{2}$$

Recall that the FL test is based on the differences between $\hat{\mu}_k$ and the full-sample average $\hat{\mu}_T$ under suitable normalization:

$$FL_T = \max_{1\le k\le T} \frac{k}{\hat{\sigma}\sqrt{T}} |\hat{\mu}_k - \hat{\mu}_T|. \tag{3}$$

For a given $h$ $(0 \le h \le 1)$, the moving estimates of $\mu_0$ are computed from windows (subsamples), each with $[Th]$ observations, moving across the whole sample. That is,

$$\tilde{\mu}_T(k,h) = \frac{1}{[Th]} \sum_{i=k+1}^{k+[Th]} y_i, \qquad k = 0, \cdots, T-[Th]. \tag{4}$$

Note that moving estimates are computed using only the most recent $[Th]$ observations in the window, and the information from the distant past (depending on the window bandwidth) is excluded sequentially as windows move forward. On the other hand, recursive estimates (2) are obtained from growing windows so that no past information is discarded.

Under the null hypothesis that the mean function is a constant, moving estimates should not fluctuate too much and should be close to $\hat{\mu}_T$. The proposed ME test for parameter stability is therefore based on the differences between $\tilde{\mu}_T(k,h)$ and $\hat{\mu}_T$. Typically, we are interested in the two-sided alternative: $\mu_i \neq \mu_0$ for some $i$. It is possible that, in some applications, we are only concerned with a one-sided alternative (e.g., $\mu_i > \mu_0$ for some $i$). This situation may arise when one has prior belief that the mean $\mu$ might have changed in only one direction or when the error of accepting the null hypothesis under $\mu_i < \mu_0$ is of no practical importance. For example, let $y$ be the ratio of defective products of a production line. A quality control manager is only interested in whether there is a significant increase of this ratio; a decrease of the defective ratio is not practically relevant. The ME statistics for one- and two-sided alternatives are, respectively,

$$ME^+_{T,h} = \max_{k=0,\cdots,T-[Th]} \frac{[Th]}{\hat{\sigma}\sqrt{T}} (\tilde{\mu}_T(k,h) - \hat{\mu}_T), \tag{5}$$

$$ME_{T,h} = \max_{k=0,\cdots,T-[Th]} \frac{[Th]}{\hat{\sigma}\sqrt{T}} |\tilde{\mu}_T(k,h) - \hat{\mu}_T|. \tag{6}$$

While the ME test is *not* motivated by any specific alternative, it can be interpreted as the maximal-LR test against the alternative of temporary parameter instability:

$$y_t = \begin{cases} \mu_1 + \epsilon_t, & t = 1, \cdots, k_1, \\ \mu_2 + \epsilon_t, & t = k_1 + 1, \cdots, k_1 + \kappa, \\ \mu_1 + \epsilon_t, & t = k_1 + \kappa + 1, \cdots, T, \end{cases} \tag{7}$$

where $\epsilon_t$ are i.i.d. $N(0,1)$, and $\kappa$ is a known number denoting the duration of parameter change. This alternative is reasonable in many applications. For example, after a policy is announced, the economy may shift to a different regime for a period of time and then return to its original level. The effectiveness of the policy is thus characterized by the duration of parameter change.

Under the null hypothesis of constant parameter, the MLE (maximum likelihood estimator) for $\mu_1$ is $\hat{\mu}_T$; under the alternative (7), the MLEs for $\mu_1$ and $\mu_2$ are

$$\hat{\mu}_{1T}(k_1,\kappa) = \frac{1}{T-\kappa}\left(\sum_{t=1}^{k_1} y_t + \sum_{t=k_1+\kappa+1}^{T} y_t\right);$$

$$\hat{\mu}_{2T}(k_1,\kappa) = \frac{1}{\kappa}\sum_{t=k_1+1}^{k_1+\kappa} y_t.$$

Let $\Lambda$ denote the likelihood ratio. The maximal-LR test statistic is

$$\max_{1<k_1<T-\kappa} 2\log\Lambda(k_1,\kappa).$$

It is not too difficult to show that (see the Appendix):

$$2\log\Lambda(k_1,\kappa) = \frac{T^2}{\kappa(T-\kappa)}\frac{\kappa^2}{T}\left(\hat{\mu}_{2T}(k_1,\kappa)-\hat{\mu}_T\right)^2. \tag{8}$$

Setting $[Th]=\kappa$, $\hat{\mu}_{2T}(k_1,\kappa)=\tilde{\mu}_T(k_1,h)$ by (4). Hence, the maximal-LR statistic, max-$LR_{T,h}$, for the alternative of a temporary change in mean of duration $[Th]$ is

$$\text{max-}LR_{T,h} = \max_{1<k_1<T-[Th]} 2\log\Lambda(k_1,[Th]) = \frac{1}{h_T(1-h_T)}ME^2_{T,h},$$

where $h_T=[Th]/T$. Equivalently,

$$ME^2_{T,h} = h_T(1-h_T)\,\text{max-}LR_{T,h}. \tag{9}$$

For a given $h$, the ME and maximal-LR tests use different normalizations which result in different asymptotic variances. Thus, these two tests are equivalent asymptotically. In finite samples, however, they may perform differently. In light of the arguments in James, James, & Siegmund [15], we would expect that the ME test has greater power than the maximal-LR test when $h$ is close to $1/2$ and that the maximal-LR test is more favorable when $h$ is close to 0 or 1. This relationship is similar to that between the FL and maximal-LR tests under a single structural change.

As the duration $[Th]$ of the parameter change is typically unknown in practice, one could of course also consider test statistics based on maximizing max-$LR_{T,h}$ over $h\in H\subseteq$

[0, 1], possibly weighted to reflect prior beliefs. Similar to the maximal-LR test under a single structural change, $H$ should be a compact subset in [0, 1] to prevent the resulting LR statistic from diverging when $h$ approaches 0 or 1; see [1] for more details. In this paper, we shall consider only fixed-$h$ ME tests for sake of simplicity. The asymptotic null distributions for tests which maximize over $h$ can be easily deduced from our results; the power of such tests is currently being investigated.

More generally, as moving estimates implement a locally weighted regression in the sense of Cleveland [10], the ME test can also be interpreted as a non-parametric test against a general non-constant mean function. By letting the bandwidth $h$ of moving windows tend to zero at a suitable rate, one can obtain a consistency result for a general, unknown regression function. Note that the FL test does not have a similar interpretation. A difficult problem is of course the determination of the optimal bandwidth $h$. Research along this direction is beyond the scope of this paper, however.

## 3 Asymptotic Null Distributions

In this section we investigate the limiting behavior of the ME test under the null hypothesis. Let $S_T$ denote the piecewise constant interpolation of

$$S_T\left(\frac{k}{T}\right) = \frac{1}{\hat{\sigma}\sqrt{T}} \sum_{i=1}^{k} \epsilon_i,$$

so that $S_T(t) = (\hat{\sigma}\sqrt{T})^{-1} \sum_{i=1}^{[Tt]} \epsilon_i$. $S_T$ is a process in $D([0,1])$, the space of functions that are right continuous with left-hand limits on $[0,1]$, and $S_T \Rightarrow W$ by the FCLT (1). Also let $S_T^0$ denote the tied-down process given by

$$S_T^0(t) = S_T(t) - \frac{[Tt]}{T} S_T(1)$$

which is also in $D([0,1])$ with jumps at $k/T$, $1 \leq k \leq T$. Note that $S_T^0(0) = S_T^0(1) = 0$ and $S_T^0 \Rightarrow W^0$, where $W^0$ is a Brownian bridge. Also note that $S_T^0$ attains its extrema in one of the jump points $k/T$. Under the null hypothesis,

$$\begin{aligned} & \frac{[Th]}{\hat{\sigma}\sqrt{T}} \left(\tilde{\mu}_T(k,h) - \hat{\mu}_T\right) \\ = & \frac{1}{\hat{\sigma}\sqrt{T}} \left( \sum_{i=1}^{k+[Th]} \epsilon_i - \sum_{i=1}^{k} \epsilon_i \right) - \frac{[Th]}{T} \frac{1}{\hat{\sigma}\sqrt{T}} \sum_{i=1}^{T} \epsilon_i \end{aligned}$$

$$
\begin{aligned}
&= S_T\left(\frac{k+[Th]}{T}\right) - S_T\left(\frac{k}{T}\right) - \frac{[Th]}{T}S_T(1) \\
&= S_T^0\left(\frac{k}{T}+h_T\right) - S_T^0\left(\frac{k}{T}\right),
\end{aligned} \tag{10}
$$

where, again, $h_T = [Th]/T$. The associated empirical ME process

$$
\begin{aligned}
M_{T,h}(t) &= \frac{[Th]}{\hat{\sigma}\sqrt{T}}(\tilde{\mu}_T([Nt],h) - \hat{\mu}_T) \\
&= S_T^0\left(\frac{[Nt]}{T}+h_T\right) - S_T^0\left(\frac{[Nt]}{T}\right), \qquad 0 \le t \le 1-h,
\end{aligned} \tag{11}
$$

is the piecewise constant interpolation on $[0, 1-h]$ of (10) with interpolation nodes $k/N$, $0 \le k \le T-[Th]$, where $N = (T-[Th])/(1-h)$. Observe that $[Nt]/T \to t$ and $h_T \to h$ as $T \to \infty$. With a little extra work, the following result follows from the continuous mapping theorem.

**Theorem 3.1** *Assume that the FCLT (1) holds. Then under the null hypothesis, if $\hat{\sigma}$ is consistent for $\sigma$, we have*

$$
M_{T,h} \Rightarrow M_h,
$$

*where $M_h(t) = W^0(t+h) - W^0(t)$ for $0 \le t \le 1-h$. In particular,*

$$
\begin{aligned}
\lim_{T\to\infty} \mathbb{P}\{ME_{T,h}^+ < \beta\} &= \mathbb{P}\{M_h(t) \le \beta \text{ for all } 0 \le t \le 1-h\}, \\
\lim_{T\to\infty} \mathbb{P}\{ME_{T,h} < \beta\} &= \mathbb{P}\{|M_h(t)| \le \beta \text{ for all } 0 \le t \le 1-h\}.
\end{aligned}
$$

This result shows that the empirical ME process converges in distribution to the increments of a Brownian bridge, and the limiting distributions of the ME tests are determined by the boundary-crossing probabilities of the limiting process $M_h$. Analytic expressions of the boundary-crossing probabilities of the $M_h$ process can be obtained from Cressie [11] for the one-sided case; for the two-sided case, they were first derived in an early version of this paper [8] for $h = 1/2$ and subsequently extended in [14] for $h \ge 1/2$. From Corollary 4 of [14] we have:

**Corollary 3.2** *Assume that the FCLT (1) holds. Then under the null hypothesis, if $\hat{\sigma}$ is consistent for $\sigma$, we have for all $\beta > 0$,*

$$
\lim_{T\to\infty} \mathbb{P}\{ME_{T,1/2}^+ < \beta\} = 2\Phi(2\beta) - 4\beta\phi(2\beta) - 1, \tag{12}
$$

*and*

$$\begin{aligned}
&\lim_{T\to\infty} \mathbb{P}\{ME_{T,1/2} < \beta\} \\
&= 1 - 4\beta \sum_{k=-\infty}^{\infty} \phi(2(2k+1)\beta) &(13)\\
&= 2\sum_{k=1}^{\infty}(-1)^{k+1} e^{-k^2\pi^2/8\beta^2}, &(14)
\end{aligned}$$

*where $\Phi$ and $\phi$ are the distribution and density functions of the standard normal random variable.*

For tail probabilities equal to 0.1, 0.05, 0.025, and 0.01, the asymptotic critical values of the one-sided ME test with $h = 1/2$ are 1.25014, 1.39774, 1.52876, and 1.68411, respectively. The critical values of the two-sided ME test are given in Table 1. For $h > 1/2$, the expressions are quite cumbersome; we omit the details. For more interesting cases $h < 1/2$ which correspond to "small" moving windows, no analytic expression is known, but the asymptotic critical values can be easily obtained by simulating the $M_h$ process.

## 4 Extension to Multiple Regression

The previous results can be extended easily to the multiple linear regression model. We now consider the model

$$y_i = x_i'\theta_i + \epsilon_i, \qquad i = 1, 2, \cdots, T,$$

where $x_i$ is a $n \times 1$ vector. The null hypothesis is that $\theta_i = \theta_0$ for all $i$.

We assume that the double array $\{x_i\epsilon_i/\sqrt{T}\}$ satisfies the conditions of Corollary 4.2 of Wooldridge & White [19] so that a multivariate FCLT holds:

$$\left(\frac{1}{\sqrt{T}}\Sigma^{-1/2}\sum_{i=1}^{[Tt]} x_i\epsilon_i, 0 \le t \le 1\right) \Rightarrow \boldsymbol{W}, \tag{15}$$

where

$$\Sigma := \lim_{T\to\infty} \frac{1}{T}\,\mathbb{E}\Big[\Big(\sum_{i=1}^{T} x_i\epsilon_i\Big)\Big(\sum_{i=1}^{T} x_i\epsilon_i\Big)'\Big],$$

and $\boldsymbol{W}$ stands for the $n$-dimensional, standard Wiener process, cf. (1). To reduce technicality, we do not state the regularity conditions explicitly, but we note that Corollary 4.2

of [19] allows $x_i$ and $\epsilon_i$ to be weakly dependent, heterogeneous random variables but *not* integrated of positive order. The limiting result in (15) again holds when $\Sigma$ is replaced by an estimator $\hat{\Sigma}_T$; for example, $\hat{\Sigma}_T$ may be a heteroskedasticity and autocorrelation consistent estimator, e.g., Newey & West [16]. In addition to the FCLT, we also assume that the following weak law of large numbers (WLLN)

$$\frac{1}{[Th]} \sum_{i=[Nt]+1}^{[Nt]+[Th]} x_i x_i' \rightarrow^p Q, \tag{16}$$

holds uniformly in $0 \leq t \leq 1-h$ for given $h$, where as usual $N = (T - [Th])/(1-h)$, $\rightarrow^p$ stands for convergence in probability and $Q$ is a non-singular, non-stochastic $n \times n$ matrix. Note that if $t = 0$ and $h = 1$, (16) gives the standard WLLN. It is not too hard to show that if

$$\frac{1}{T} \sum_{i=1}^{T} x_i x_i' \rightarrow Q$$

almost surely, then (16) holds uniformly in $0 \leq t \leq 1-h$. We omit the details.

For a given $h$, the moving OLS estimates are

$$\tilde{\theta}_T(k,h) = \left( \sum_{i=k+1}^{k+[Th]} x_i x_i' \right)^{-1} \left( \sum_{i=k+1}^{k+[Th]} x_i y_i \right), \quad k = 0, \cdots, T-[Th]. \tag{17}$$

Let $\hat{\theta}_T$ be the standard (full-sample) OLS estimator, $Q_T = T^{-1} \sum_{i=1}^{T} x_i x_i'$, and $\hat{D}_T = Q_T^{-1} \hat{\Sigma}_T Q_T^{-1}$ such that $\hat{D}_T^{-1/2} = \hat{\Sigma}_T^{-1/2} Q_T$. Here, $\hat{\Sigma}_T$ is again an estimator of $\Sigma$. For an $n$-dimensional vector $V$, let $||V|| = \max_{j=1,\cdots,n} |V_j|$ be the maximum norm of $V$, where $V_j$ is the $j$-th element of $V$. The ME test statistic for the multiple regression model is

$$ME_{T,h} = \max_{k=0,\cdots,T-[Th]} \frac{[Th]}{\sqrt{T}} ||\hat{D}_T^{-1/2}(\tilde{\theta}_T(k,h) - \hat{\theta}_T)||. \tag{18}$$

In what follows, notations in boldface are the multivariate analogues of those used in Section 2. Let

$$\mathbf{S}_T(t) = \frac{1}{\sqrt{T}} \hat{\Sigma}_T^{-1/2} \sum_{i=1}^{[Tt]} x_i \epsilon_i$$

and let $\mathbf{S}_T^0$ denote the tied-down process given by

$$\mathbf{S}_T^0(t) = \mathbf{S}_T(t) - \frac{[Tt]}{T} \mathbf{S}_T(1).$$

Under the null hypothesis, we have

$$
\begin{aligned}
&\frac{[Th]}{\sqrt{T}}\,\hat{D}_T^{-1/2}\,(\tilde{\theta}_T(k,h)-\hat{\theta}_T) \\
&= \frac{1}{\sqrt{T}}\,\hat{D}_T^{-1/2}\left(\frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i x_i'\right)^{-1}\left(\sum_{i=1}^{k+[Th]} x_i\epsilon_i - \sum_{i=1}^{k} x_i\epsilon_i\right) \\
&\quad - \frac{[Th]}{T}\,\hat{D}_T^{-1/2}\left(\frac{1}{T}\sum_{i=1}^{T} x_i x_i'\right)^{-1}\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T} x_i\epsilon_i\right) \\
&= \frac{1}{\sqrt{T}}\,\hat{\Sigma}_T^{-1/2}\left[Q_T\left(\frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i x_i'\right)^{-1} - I\right]\left(\sum_{i=1}^{k+[Th]} x_i\epsilon_i - \sum_{i=1}^{k} x_i\epsilon_i\right) \\
&\quad + S_T\left(\frac{k+[Th]}{T}\right) - S_T\left(\frac{k}{T}\right) - \frac{[Th]}{T}S_T(1). \qquad (19)
\end{aligned}
$$

It is readily seen that the first term on the right-hand side of (19) is $o_p(1)$ under the WLLN (16) and that the sum of the last three terms is $S_T^0(k/T+h_T)-S_T^0(k/T)$. Let the multivariate empirical ME process $M_{T,h}$ be the piecewise constant interpolation of (19), cf. (11), and let $W^0$ be an $n$-dimensional Brownian bridge. The following result is a multivariate extension of Theorem 3.1.

**Theorem 4.1** *Assume that the FCLT* (15) *and WLLN* (16) *hold. Then under the null hypothesis, if $\hat{\Sigma}_T$ is consistent for $\Sigma$, we have*

$$M_{T,h} \Rightarrow M_h,$$

*where $M_h(t)=W^0(t+h)-W^0(t)$ for $0\le t\le 1-h$. In particular,*

$$
\begin{aligned}
&\lim_{T\to\infty} \mathbb{P}\{ME_{T,h}\le\beta\} \\
&= \mathbb{P}\{\|M_h(t)\|\le\beta \text{ for all } 0\le t\le 1-h\} \\
&= \left(\mathbb{P}\{|M_h(t)|\le\beta \text{ for all } 0\le t\le 1-h\}\right)^n,
\end{aligned}
$$

*where $M_h(t)=W^0(t+h)-W^0(t)$, and $n$ is the number of parameters in the model.*

Hence, only asymptotic critical values for $n=1$ need to be tabulated. For $h=1/2$, we obtain from Corollary 3.2 that:

**Corollary 4.2** *Assume that the FCLT* (15) *and WLLN* (16) *hold. Then under the null hypothesis, if $\hat{\Sigma}_T$ is consistent for $\Sigma$, we have*

$$\lim_{T\to\infty} \mathbb{P}\{ME_{T,1/2}\le\beta\} = \left(2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k^2\pi^2/8\beta^2}\right)^n.$$

For $h = 1/2$, we summarize the asymptotic critical values for $n = 1, \ldots, 10$ in Table 1 for completeness. For some selected $h < 1/2$, simulated asymptotic critical values are tabulated in Table 2 for $n = 1$ only; a complete table is available upon request. In our simulation, the Wiener process is approximated by a discrete-time random walk driven by the Gaussian innovations with 2000 observations, and the number of replications is 100000. For $h = 1/2$, simulated critical values are in fact very close to those in Table 1.

[ Tables 1 and 2 About Here ]

We now consider a general class of alternatives:

$$\theta_i = \theta_0 + T^{-\delta} g(i/T), \tag{20}$$

where $g\colon [0,1] \to \mathbb{R}^n$ is a (non-constant) vector-valued function of bounded variation on $[0,1]$. If $\delta = 0$, (20) is a global alternative; if $\delta = 1/2$, (20) characterizes a sequence of local alternatives. $g$ may e.g. be a step function to represent multiple structural changes or a continuous function to represent smooth or periodic parameter changes.

**Theorem 4.3** *Assume that the FCLT* (15) *and WLLN* (16) *hold. Then under the alternative* (20), *if* $\hat{\Sigma}_T^{-1}$ *is* $O_p(1)$, *we have*

$$T^{\delta-1/2} ME_{T,h} = \max_{0 \le t \le 1-h} \left\| \hat{\Sigma}_T^{-1/2} (Q L_h g(t) + T^{\delta-1/2} \Sigma^{1/2} M_{T,h}(t)) \right\| + o_p(1), \tag{21}$$

*where*

$$L_h g(t) = \int_t^{t+h} g(u)\, du - h \int_0^1 g(u)\, du.$$

Notice that under the alternative, $\hat{\Sigma}_T$ is not necessarily consistent for $\Sigma$; in particular, under global alternatives ($\delta = 0$), it is not consistent in standard cases (see further below).

As the derivative of the function $L_h g$ at $t$ is $g(t+h) - g(t)$, $L_h g$ is nonzero provided that $g$ is not periodic with period $h$. Conversely, if $g$ has period $h$, and $1/h$ is an integer, then

$$L_h g(t) = \int_t^{t+h} g(u)\, du - h \int_0^1 g(u)\, du = \int_0^h g(u)\, du - h \frac{1}{h} \int_0^h g(u)\, du = 0.$$

Hence, if in addition to the conditions of theorem 4.3, $\hat{\Sigma}_T$ is also $O_p(1)$, $0 \le \delta < 1/2$ and $g$ is not periodic with period $h$, the right-hand side of (21) is bounded away from zero (in probability), and the ME test statistic grows at rate $T^{1/2-\delta}$; therefore, the ME test is consistent against such sequences of alternatives. On the other hand, the test is not consistent if $g$ has period $h$ and $1/h$ is an integer.

More definite results on the asymptotics of the ME test under alternatives (20) can be given under suitable assumptions on the structure of $\Sigma$. Suppose that $\{x_t\}$ is a sequence of suitably mixing random variables and that $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with mean zero and variance $\sigma^2$ which is independent of $\{x_t\}$, as assumed in [17]. Given these conditions, $\Sigma = \sigma^2 Q$, and a natural estimate is

$$\hat{\Sigma}_T = \hat{\sigma}^2 Q_T, \qquad \hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{T}(y_i - x_i'\hat{\theta}_T)^2. \tag{22}$$

It can easily be shown that

$$\hat{\sigma}^2 \to^p \sigma_\delta^2 = \begin{cases} \sigma^2, & 0 < \delta \le \frac{1}{2}, \\ \sigma^2 + \int_0^1 \left(g(u) - \int_0^1 g(v)\,dv\right)' Q \left(g(u) - \int_0^1 g(v)\,dv\right) du, & \delta = 0; \end{cases}$$

notice that $\sigma_0^2 > \sigma^2$ unless $g$ is constant. Hence, we have the following.

**Corollary 4.4** *Assume that the conditions in* [17 p. 308] *hold and that* $\hat{\Sigma}_T$ *is given by* (22). *Then under the alternatives* (20) *with* $\delta = 1/2$,

$$ME_{T,h} \Rightarrow \max_{0\le t\le 1-h} \left\| M_h(t) + \sigma^{-1} Q^{1/2} L_h g(t) \right\|;$$

*if* $0 \le \delta < 1/2$,

$$T^{\delta - 1/2} ME_{T,h} \to^p \sigma_\delta^{-1} \max_{0\le t\le 1-h} \|Q^{1/2} L_h g(t)\|.$$

**Remark:** In view of the remarks after Theorem 4.3, if $\delta = 1/2$ and $g$ is periodic with period $h$ and $1/h$ is an integer,

$$ME_{T,h} \Rightarrow \max_{0\le t\le 1-h} \|M_h(t)\|.$$

which is identical to the limit under the null hypothesis. Thus, the ME test has only trivial local power against such alternatives.

## 5 Test Performance and Simulation

In this section we report some simulation results. For the simulation of empirical test sizes, the data $y_i$ are generated from i.i.d. $N(2,1)$, the number of replications is 10,000, and the variance estimate of the FL and ME tests is the standard one:

$$\hat{\sigma}^2 = \sum_{i=1}^{T} (y_i - \hat{\mu}_T)^2 / T,$$

where $\hat{\mu}_T$ is the full-sample average. For samples $T = 100, 200, 300$, and 500, the resulting empirical sizes of the ME test ($h = 1/2$) at 10% level are 7%, 7.5%, 8.2%, and 9.2%, respectively. They are quite reasonable and similar to the empirical sizes of the FL test.

For the simulation of power performance, we consider the ME tests with $h = 1/2$, $h = 1/5$, and $h = 1/10$. (We shall write ME($h$) to indicate the dependence on $h$.) The competing tests are the FL, maximal-F (MAX-F), average-F (AVG-F), and (average) exponential-F (EXP-F) tests, where the AVG-F and EXP-F tests are asymptotically optimal in the sense of Andrews & Ploberger [2]. All power simulations are based on sample size $T = 100$ and the number of replications 2500. Empirical critical values are obtained from simulations with 10000 replications: ME(1/2) = 1.289, ME(1/5) = 1.149, ME(1/10) = 0.91, FL = 1.176. For the alternative of a single structural change (23), the set of hypothetical change points $[Tc]$ for the MAX-F, AVG-F, and EXP-F test is such that $c \in [0.1, 0.9]$; see [1]. The empirical critical values are MAX-F = 7.328, AVG-F = 2.157, EXP-F = 1.6. It can be verified that these values are quite close to the corresponding asymptotic critical values. For the alternative of double structural changes (24), the set of hypothetical change points $[Tc_1]$ and $[Tc_2]$ is such that

$$\{(c_1, c_2) : \ c_1 = i/20, c_2 = j/20, \ \ 2 \le i < j \le 18\}$$

with mesh 1/20. Thus, tests are computed as there are 136 possible "double breaks". The empirical critical values are: MAX-F = 5.718, AVG-F = 1.861, and EXP-F = 2.756. We note that the performance of these tests may be affected by the choice of the indices $(c_1, c_2)$. Our simulation is designed to permit a "fair" comparison, as the set of indices is chosen to be compatible with the way we generate (24).

The data generating process (DGP) for a single structural change is

$$y_i = \begin{cases} 2 + \epsilon_i, & i = 1, \cdots, [T\lambda], \\ 2 + \Delta + \epsilon_i, & i = [T\lambda] + 1, \cdots, T, \end{cases} \tag{23}$$

where $\Delta = 0.4$ and $\epsilon_i$ are i.i.d. $N(0,1)$. We consider $\lambda = 0.1, 0.2, \cdots, 0.9$. Because the power performance of these tests is symmetric in $\lambda$, we only report the results for $\lambda$ from 0.1 to 0.5 in Table 3. It can be seen that the FL, MAX-F, AVG-F, and EXP-F tests do not dominate each other. In particular, the FL, AVG-F, EXP-F are the best for $\lambda \geq 0.3$, and the MAX-F and EXP-F are the best for $\lambda \leq 0.2$. The ME tests are comparable with the FL test when $\lambda = 0.1$; they are, however, dominated by the three $F$ tests for all $\lambda$.

[ Table 3 About Here ]

For the alternative of temporary parameter instability, we first consider the DGP:

$$y_i = \begin{cases} 2 + \epsilon_i, & i = 1, \cdots, [T\lambda_1], \\ 2 + \Delta + \epsilon_i, & i = [T\lambda_1] + 1, \cdots, [T\lambda_2], \\ 2 + \epsilon_i, & i = [T\lambda_2] + 1, \cdots, T, \end{cases} \tag{24}$$

with $\Delta = 0.4$. We consider the change points $\lambda_1 = 0.2, \ldots, 0.8$ and $\lambda_2 = \lambda_1 + 0.1, \ldots, 0.9$. We also compute the MAX-F, AVG-F, and EXP-F tests under a single break alternative to see what would happen if "incorrect" tests are used. The empirical power results are collected in Table 4. It can be seen that the FL test and tests for a single change have relatively low power. When two change points differ by 0.3 or larger, the ME(1/2) test performs better than the MAX-F, AVG-F, and EXP-F tests for double changes and has the highest power when the difference between two break points is 1/2, the bandwidth of moving windows. When two break points are close, the ME(1/2) test performs similarly to the AVG-F and EXP-F tests, When two break points differ by 0.2, the ME(1/5) is the best, but the MAX-F test is comparable. When two break points differ by 0.1, the MAX-F test is usually the best, but the ME(1/5) and ME(1/10) tests are comparable. This result agrees with the relationship between the ME and maximal-LR tests discussed in section 2.

We also consider a DGP in which the parameter has a sudden jump first but gradually returns to the original level:

$$y_i = \begin{cases} 2 + \epsilon_i, & i = 1, \cdots, [T\lambda_1], \\ 2 + \Delta \left( \dfrac{[T\lambda_2] - i + 1}{[T\lambda_2] - [T\lambda_1]} \right) + \epsilon_i, & i = [T\lambda_1] + 1, \cdots, [T\lambda_2], \\ 2 + \epsilon_i, & i = [T\lambda_2] + 1, \cdots, T, \end{cases} \tag{25}$$

with $\Delta = 0.4$. In this case, we consider $\lambda_1 = 0.2, 0.4, 0.6$, and $0.8$ and $\lambda_2 = \lambda_1 + 0.2, \ldots, 1$. These results are summarized in Table 5. It can be seen that the ME(1/2) test performs slightly better than the MAX-F, AVG-F, and EXP-F tests for double changes when $\lambda_2 - \lambda_1 \geq 0.4$. When two break points are close, these four tests and the ME(1/5) test are quite similar. Again, the FL test and tests for a single change have relatively lower power.

[ Tables 4 and 5 About Here ]

# 6 Summary

In this paper we propose the ME test for parameter stability and analyze its limiting behavior under the null and alternative hypotheses. We tabulate some asymptotic critical values for the ME tests with various moving-window bandwidths and show that this class of tests is consistent and has non-trivial local power in general. It is also shown by simulations that when there are double structural changes, the ME tests with different $h$ are superior to or comparable with other competing tests, including the AVG-F and EXP-F tests which are optimal in the sense of [2]. Thus, the ME tests can complement other tests for parameter stability in the literature. There are some limitations of our results, however. First, our results do not apply to models with integrated and trending regressors. Second, the optimal moving-window bandwidth remains unknown. As the ME test can be interpreted as a non-parametric test for a general non-constant mean function, the second problem may be tackled from a non-parametric regression perspective. This issue is currently under investigation.

## Appendix

**Proof of Equation (8):** The likelihood ratio is

$$\begin{aligned}
&2\log\Lambda(k_1,\kappa)\\
&= \sum_{t=1}^{T}(y_t-\hat{\mu}_T)^2 - \left(\sum_{t=1}^{k_1}(y_t-\hat{\mu}_{1T})^2 + \sum_{t=k_1+1}^{k_1+\kappa}(y_t-\hat{\mu}_{2T})^2 + \sum_{t=k_1+\kappa+1}^{T}(y_t-\hat{\mu}_{1T})^2\right)\\
&= (T-\kappa)\hat{\mu}_{1T}^2 + \kappa\hat{\mu}_{2T}^2 - T\hat{\mu}_T^2\\
&= (T-\kappa)\hat{\mu}_{1T}^2 + \kappa\hat{\mu}_{2T}^2 - T\hat{\mu}_T^2.
\end{aligned}$$

As $T\hat{\mu}_T = (T-\kappa)\hat{\mu}_{1T} + \kappa\hat{\mu}_{2T}$, we have

$$(T-\kappa)\hat{\mu}_{1T}^2 = \frac{1}{T-\kappa}\Big(T^2\hat{\mu}_T^2 - 2T\kappa\hat{\mu}_T\hat{\mu}_{2T} + \kappa^2\hat{\mu}_{2T}^2\Big).$$

It follows that

$$\begin{aligned}
2\log\Lambda(k_1,\kappa) &= \frac{1}{T-\kappa}\Big(T^2\hat{\mu}_T^2 - 2T\kappa\hat{\mu}_T\hat{\mu}_{2T} + \kappa^2\hat{\mu}_{2T}^2\Big) + \kappa\hat{\mu}_{2T}^2 - T\hat{\mu}_T^2\\
&= \frac{T\kappa}{T-\kappa}\Big(\hat{\mu}_T^2 + \hat{\mu}_{2T}^2 - 2\hat{\mu}_T\hat{\mu}_{2T}\Big)\\
&= \frac{T\kappa}{T-\kappa}\Big(\hat{\mu}_{2T} - \hat{\mu}_T\Big)^2\\
&= \frac{T^2}{\kappa(T-\kappa)}\frac{\kappa^2}{T}\Big(\hat{\mu}_{2T} - \hat{\mu}_T\Big)^2. \quad \square
\end{aligned}$$

**Lemma A1** *Let $X_T$ be a sequence of random processes in $D([0,1])^k$ converging in distribution (with respect to the Skorohod topology) to a random process $X$ in $C([0,1])^k$ (i.e., the limiting process has continuous paths). Further, let $0 < h_T < 1$ be a sequence converging to $0 < h < 1$, and let $\kappa_T : [0, 1-h] \to [0, 1-h_T]$ be a sequence of maps such that $\sup_{0\le t\le 1-h}|\kappa_T(t) - t|$ tends to zero. Then, if $Z_T$ is the random process on $D([0,1-h])^k$ given by*

$$Z_T(t) = X_T(\kappa_T(t) + h_T) - X_T(\kappa_T(t)),$$

*we have $Z_T \Rightarrow Z$, where for $0 \le t \le 1-h$, $Z(t) = X(t+h) - X(t)$.*

**Proof:** For a function $f$ in $D([0,1])^k$, put

$$\omega_f(\delta) = \max\nolimits_{0\le s,t\le 1;|s-t|\le\delta}|f(s) - f(t)|.$$

Also, let $\delta_T = \max\left(|h_T - h|, \sup_{0\le t\le 1-h}|\kappa_T(t) - t|\right)$ and let $Z_T^*(t) = X_T(t+h) - X_T(t)$. Then $\delta_T \to 0$ and the continuous mapping theorem yields that $Z_T^* \Rightarrow Z$. As

$$\begin{aligned} & |Z_T(t) - Z_T^*(t)| \\ & \quad \le \ |X_T(\kappa_T(t) + h_T) - X_T(t+h)| + |X_T(\kappa_T(t)) - X_T(t)| \\ & \quad \le \ 2\,\omega_{X_T}(2\delta_T), \end{aligned}$$

we have $\sup_{0\le t\le 1-h}|Z_T(t) - Z_T^*(t)| \le 2\omega_{X_T}(2\delta_T) = o_p(1)$ because $X_T \Rightarrow X$ and $X$ has continuous sample paths. It follows from Theorem 4.1 of Billingsley [5] that $Z_T$ has the same weak limit as $Z_T^*$, whence $Z_T \Rightarrow Z$. □

**Proof of Theorem 3.1:** We apply Lemma A1. Setting $X_T = S_T^0$, $X = W^0$, and $\kappa_T(t) = [Nt]/T$. Note that for $0 \le t \le 1-h$, $0 \le \kappa_T(t) \le 1 - h_T$, where $h_T = [Th]/T$. Clearly, $\sup_{0\le t\le 1-h}|\kappa_T(t) - t| \to 0$. It follows from (11) and Lemma A1 that $M_{T,h} \Rightarrow M_h$. Since $M_{T,h}$ is piecewise constant, it reaches its extrema at one of the jumps. Thus, from (6) we have

$$\begin{aligned} ME_{T,h}^+ &= \max_{0\le t\le 1-h} M_{T,h}(t) \Rightarrow \max_{0\le t\le 1-h} M_h(t), \\ ME_{T,h} &= \max_{0\le t\le 1-h} |M_{T,h}(t)| \Rightarrow \max_{0\le t\le 1-h} |M_h(t)|, \end{aligned}$$

by the continuous mapping theorem. □

**Proof of Corollary 3.2:** Let $(X(t), 0 \le t \le 1)$ be a zero mean, continuous-path stationary Gaussian process with

$$\mathrm{cov}(X(t_1), X(t_2)) = 1 - \kappa|t_1 - t_2|.$$

Hornik [13, Corollary 4] proves that for $\kappa = 2$ and $b > 0$,

$$\mathbb{P}\{\max_{0\le t\le 1} X(t) \le b\} = 2\Phi(b) - 2b\phi(b) - 1.$$

Note that $M_h$ is a zero mean, continuous-path stationary Gaussian process on $[0, 1-h]$ with

$$\mathrm{cov}(M_h(t_1), M_h(t_2)) = \sigma_h^2 - \min(h, |t_1 - t_2|),$$

where $\sigma_h^2 = h(1-h)$, so that the covariance function of $M_{1/2}$ is $1/4 - |t_1 - t_2|$. Thus, $2(M_{1/2}(\tau/2), 0 \le \tau \le 1)$ has the covariance function $1 - 2|\tau_1 - \tau_2|$. It follows that

$$\mathbb{P}\{\max_{0\le t\le 1/2} M_{1/2}(t) \le \beta\}$$

$$= \mathbb{P}\{\max_{0\le\tau\le1} 2M_{1/2}(\tau/2) \le 2\beta\}$$
$$= \mathbb{P}\{\max_{0\le\tau\le1} X(\tau) \le 2\beta\}$$
$$= 2\Phi(2\beta) - 4\beta\phi(2\beta) - 1.$$

The second assertion follows similarly. We note that these distributions can be derived using a different approach; details are in the early version of this paper, [8]. □

**Proof of Theorem 4.1:** As the square bracket term in (19):

$$Q_T\left(\frac{1}{[Th]}\sum_{i=[Nt]+1}^{[Nt]+[Th]} x_i x_i'\right)^{-1} - I$$

converges to zero in probability uniformly in $t$ by the WLLN (16),

$$\boldsymbol{M}_{T,h}(t) - (\boldsymbol{S}_T^0([Nt]/T + h_T) - \boldsymbol{S}_T^0([Nt]/T)) \to^p 0$$

uniformly in $t$. Hence, it suffices to consider the interpolation of $\boldsymbol{S}_T^0([Nt]/T + h_T) - \boldsymbol{S}_T^0([Nt]/T)$ for $0 \le t \le 1-h$ as in (11). The proof of $\boldsymbol{M}_{T,h} \Rightarrow \boldsymbol{M}_h$ and the first equality in the second assertion is therefore the same as the proof of Theorem 3.1. To prove the second equality, note that the elements $W_j^0$, $j = 1, \ldots, n$ of $\boldsymbol{W}^0$ are mutually independent, univariate Brownian bridges. Thus,

$$\lim_{T\to\infty} \mathbb{P}\{ME_{T,h} \le \beta\}$$
$$= \mathbb{P}\{\|\boldsymbol{W}^0(t+h) - \boldsymbol{W}^0(t)\| \le \beta \text{ for all } 0 \le t \le 1-h\}$$
$$= \mathbb{P}\{|W_j^0(t+h) - W_j^0(t)| \le \beta \text{ for all } j = 1, \cdots, n,\ 0 \le t \le 1-h\}$$
$$= \Big(\mathbb{P}\{|M_h(t)| \le \beta \text{ for all } 0 \le t \le 1-h\}\Big)^n. \quad \square$$

**Proof of Corollary 4.2:** Straightforward from Theorem 4.1 and Corollary 3.2. □

**Proof of Theorem 4.3:** We use the following notations:

$$R_{k,T} = \sum_{i=k+1}^{k+[Th]} x_i x_i' g(i/T), \qquad R_T = \sum_{i=1}^{T} x_i x_i' g(i/T),$$
$$Q_{k,T} = \frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i x_i'.$$

Under the alternative (20),

$$\begin{aligned}
\hat{\theta}_T &= \theta_0 + \left(\frac{1}{T}\sum_{i=1}^{T} x_i x_i'\right)^{-1} \left(\frac{1}{T^{1+\delta}}\sum_{i=1}^{T} x_i x_i' g(i/T) + \frac{1}{T}\sum_{i=1}^{T} x_i \epsilon_i\right) \\
&= \theta_0 + Q_T^{-1}\left(\frac{1}{T^{1+\delta}} R_T + \frac{1}{T}\sum_{i=1}^{T} x_i \epsilon_i\right),
\end{aligned}$$

and the moving OLS estimates are

$$\begin{aligned}
\tilde{\theta}_T(k,h) &= \theta_0 + \left(\frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i x_i'\right)^{-1} \\
&\qquad \left(\frac{1}{T^{1+\delta} h_T}\sum_{i=k+1}^{k+[Th]} x_i x_i' g(i/T) + \frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i \epsilon_i\right) \\
&= \theta_0 + Q_{k,T}^{-1}\left(\frac{1}{T^{1+\delta} h_T} R_{k,T} + \frac{1}{[Th]}\sum_{i=k+1}^{k+[Th]} x_i \epsilon_i\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&T^{\delta-1/2} ME_{T,h} \\
&= \max_{0\le t\le 1-h} T^{\delta} h_T \|\hat{D}_T^{-1/2}(\tilde{\theta}_T([Nt],h) - \hat{\theta}_T)\| \\
&= \max_{0\le t\le 1-h} \Bigg\| \hat{\Sigma}_T^{-1/2} Q_T \Bigg[ Q_{[Nt],T}^{-1}\left(\frac{1}{T} R_{[Nt],T} + T^{\delta-1/2}\frac{1}{\sqrt{T}}\sum_{i=[Nt]+1}^{[Nt]+[Th]} x_i \epsilon_i\right) \\
&\qquad - Q_T^{-1}\left(h_T \frac{1}{T} R_T + T^{\delta-1/2} h_T \frac{1}{\sqrt{T}}\sum_{i=1}^{T} x_i \epsilon_i\right)\Bigg]\Bigg\|.
\end{aligned}$$

Provided the WLLN (16) holds and that $g$ is of bounded variation on $[0,1]$, it can be shown that

$$\frac{1}{T} R_{[Nt],T} = \frac{1}{T}\sum_{i=[Nt]+1}^{[Nt]+[Th]} x_i x_i' g(i/T) \to^p Q \int_t^{t+h} g(u)\,du$$

uniformly on $[0, 1-h]$ and that

$$\frac{1}{T} R_T = \frac{1}{T}\sum_{i=1}^{T} x_i x_i' g(i/T) \to^p Q \int_0^1 g(u)\,du.$$

Hence,

$$\hat{\Sigma}_T^{-1/2} Q_T \left[ Q_{[Nt],T}^{-1}\Big(\frac{1}{T} R_{[Nt],T}\Big) - Q_T^{-1} h_T \Big(\frac{1}{T} R_T\Big)\right]$$

$$
\begin{aligned}
&= \hat{\Sigma}_T^{-1/2} Q \left[ Q^{-1} Q \int_t^{t+h} g(u)\, du - h Q^{-1} Q \int_0^1 g(u)\, du \right] + o_p(1) \\
&= \hat{\Sigma}_T^{-1/2} Q L_h g(t) + o_p(1)
\end{aligned}
$$

uniformly on $[0, 1-h]$. As clearly

$$
Q_T Q_{[Nt],T}^{-1} \frac{1}{\sqrt{T}} \sum_{i=[Nt]+1}^{[Nt]+[Th]} x_i \epsilon_i - h_T \frac{1}{\sqrt{T}} \sum_{i=1}^{T} x_i \epsilon_i = \Sigma^{1/2} M_{T,h}(t) + o_p(1)
$$

uniformly on $[0, 1-h]$ and $\hat{\Sigma}_T^{-1}$ is $O_p(1)$ by assumption, we obtain

$$
\begin{aligned}
& T^{\delta-1/2} ME_{T,h} \\
&= \max_{0 \le t \le 1-h} \left\| \hat{\Sigma}_T^{-1/2} \Big( Q L_h g(t) + T^{\delta-1/2} (\Sigma^{1/2} M_{T,h}(t) + o_p(1)) \Big) \right\| \\
&= \max_{0 \le t \le 1-h} \left\| \hat{\Sigma}_T^{-1/2} (Q L_h g(t) + T^{\delta-1/2} \Sigma^{1/2} M_{T,h}(t)) \right\| + o_p(1). \quad \square
\end{aligned}
$$

**Proof of Corollary 4.4:** The conclusions follow easily from Theorem 4.3 and (22). □

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**Table 1.** The Asymptotic Critical Values of the ME Test with $h = 1/2$.

| $n$ | Tail Probability | | | |
|---|---|---|---|---|
| | 0.10 | 0.05 | 0.025 | 0.01 |
| 1 | 1.37506 | 1.51151 | 1.63408 | 1.78082 |
| 2 | 1.50667 | 1.63193 | 1.74546 | 1.88269 |
| 3 | 1.57852 | 1.69814 | 1.80711 | 1.93951 |
| 4 | 1.62747 | 1.74345 | 1.84947 | 1.97871 |
| 5 | 1.66437 | 1.77772 | 1.88160 | 2.00854 |
| 6 | 1.69387 | 1.80519 | 1.90740 | 2.03255 |
| 7 | 1.71838 | 1.82805 | 1.92891 | 2.05261 |
| 8 | 1.73931 | 1.84760 | 1.94734 | 2.06980 |
| 9 | 1.75753 | 1.86465 | 1.96342 | 2.08483 |
| 10 | 1.77366 | 1.87976 | 1.97769 | 2.09819 |

*Note:* The critical values are solved numerically from the formula of Corollary 4.2 with 10 terms in the summation; $n$ is the number of parameters in the model.

**Table 2.** The Simulated Asymptotic Critical Values of the ME Tests for $n = 1$.

| $h$ | Tail Probability | | | |
|---|---|---|---|---|
| | 0.10 | 0.05 | 0.025 | 0.01 |
| 0.05 | 0.7558 | 0.8018 | 0.8437 | 0.8986 |
| 0.10 | 0.9812 | 1.0475 | 1.1064 | 1.1889 |
| 0.15 | 1.1218 | 1.2069 | 1.2861 | 1.3809 |
| 0.20 | 1.2206 | 1.3186 | 1.4080 | 1.5163 |
| 0.25 | 1.2812 | 1.3990 | 1.4991 | 1.6132 |
| 0.30 | 1.3242 | 1.4471 | 1.5597 | 1.6875 |
| 0.35 | 1.3515 | 1.4821 | 1.5994 | 1.7437 |
| 0.40 | 1.3603 | 1.4946 | 1.6171 | 1.7575 |
| 0.45 | 1.3566 | 1.5034 | 1.6209 | 1.7846 |
| 0.50 | 1.3542 | 1.4961 | 1.6231 | 1.7711 |

**Table 3.** Power Simulation under a Single Structural Change: DGP (23).

| $\lambda$ | ME Tests | | | FL | Tests for a Single Change | | |
|---|---|---|---|---|---|---|---|
| | $h = \frac{1}{2}$ | $h = \frac{1}{5}$ | $h = \frac{1}{10}$ | Test | MAX-F | AVG-F | EXP-F |
| 0.1 | 13.6 | 12.2 | 14.2 | 13.7 | 19.4 | 15.3 | 18.0 |
| 0.2 | 21.2 | 23.0 | 20.1 | 27.2 | 32.4 | 31.1 | 32.5 |
| 0.3 | 31.9 | 32.0 | 22.8 | 42.1 | 40.7 | 44.2 | 43.3 |
| 0.4 | 39.9 | 32.8 | 23.4 | 51.8 | 45.4 | 50.8 | 49.4 |
| 0.5 | 46.9 | 35.2 | 24.2 | 55.5 | 49.2 | 54.0 | 52.8 |

**Table 4.** Power Simulation under Double Structural Changes: DGP (24).

| $\lambda_1$ | $\lambda_2$ | ME Tests | | | FL | Tests for Double Changes | | | Tests for a Single Change | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $h=\frac{1}{2}$ | $h=\frac{1}{5}$ | $h=\frac{1}{10}$ | Test | MAX-F | AVG-F | EXP-F | MAX-F | AVG-F | EXP-F |
| 0.2 | 0.3 | 14.2 | 14.2 | 14.8 | 13.7 | 16.7 | 14.3 | 15.5 | 13.5 | 13.2 | 13.2 |
| | 0.4 | 22.4 | 26.0 | 20.5 | 21.0 | 24.8 | 21.4 | 24.9 | 19.0 | 17.6 | 18.8 |
| | 0.5 | 32.2 | 32.3 | 22.7 | 25.1 | 29.6 | 28.6 | 30.4 | 22.5 | 21.3 | 22.7 |
| | 0.6 | 42.6 | 32.0 | 23.2 | 26.2 | 34.5 | 33.3 | 36.3 | 25.4 | 23.0 | 25.2 |
| | 0.7 | 46.6 | 31.3 | 23.1 | 23.9 | 35.2 | 34.6 | 37.6 | 28.3 | 22.8 | 27.4 |
| | 0.8 | 40.1 | 27.6 | 21.3 | 20.4 | 34.8 | 35.5 | 38.0 | 30.4 | 21.4 | 27.5 |
| | 0.9 | 32.6 | 22.8 | 20.2 | 18.8 | 31.5 | 31.0 | 33.5 | 28.0 | 20.6 | 25.9 |
| 0.3 | 0.4 | 12.0 | 14.8 | 14.9 | 11.6 | 15.6 | 11.8 | 14.8 | 11.4 | 10.4 | 11.2 |
| | 0.5 | 21.7 | 26.2 | 20.9 | 17.8 | 24.4 | 19.0 | 22.9 | 15.8 | 13.5 | 15.0 |
| | 0.6 | 31.3 | 32.5 | 23.8 | 21.3 | 30.6 | 27.0 | 31.0 | 20.0 | 16.8 | 19.4 |
| | 0.7 | 42.5 | 33.8 | 24.8 | 23.2 | 35.4 | 33.4 | 36.4 | 24.4 | 20.1 | 23.3 |
| | 0.8 | 47.3 | 30.8 | 22.0 | 24.1 | 34.6 | 36.4 | 37.6 | 26.8 | 21.9 | 26.1 |
| | 0.9 | 41.1 | 28.4 | 20.3 | 29.6 | 34.1 | 36.4 | 36.9 | 31.8 | 28.0 | 31.4 |
| 0.4 | 0.5 | 12.2 | 14.2 | 14.1 | 10.9 | 15.0 | 10.8 | 13.4 | 11.1 | 9.1 | 10.6 |
| | 0.6 | 21.9 | 26.0 | 20.2 | 15.7 | 23.3 | 18.6 | 22.8 | 15.8 | 12.0 | 14.8 |
| | 0.7 | 31.5 | 29.9 | 22.5 | 21.8 | 31.3 | 26.4 | 31.6 | 21.8 | 17.6 | 20.6 |
| | 0.8 | 41.3 | 32.2 | 24.1 | 26.0 | 34.4 | 33.4 | 36.7 | 26.2 | 22.6 | 25.2 |
| | 0.9 | 46.7 | 31.5 | 22.9 | 36.9 | 37.5 | 40.2 | 41.2 | 34.3 | 34.2 | 35.2 |
| 0.5 | 0.6 | 12.7 | 15.0 | 16.0 | 11.4 | 14.6 | 11.4 | 13.0 | 10.6 | 9.5 | 10.6 |
| | 0.7 | 21.7 | 24.7 | 19.4 | 17.0 | 21.4 | 18.2 | 21.2 | 15.7 | 13.4 | 14.9 |
| | 0.8 | 31.5 | 31.0 | 22.5 | 24.5 | 30.8 | 27.4 | 30.8 | 21.8 | 20.6 | 21.6 |
| | 0.9 | 41.2 | 31.0 | 23.4 | 36.6 | 34.4 | 36.2 | 37.0 | 30.8 | 32.1 | 32.5 |
| 0.6 | 0.7 | 12.8 | 14.8 | 14.8 | 12.0 | 16.0 | 12.1 | 14.6 | 12.2 | 10.8 | 11.2 |
| | 0.8 | 22.2 | 27.2 | 21.9 | 19.8 | 25.3 | 20.2 | 23.8 | 17.8 | 16.9 | 17.9 |
| | 0.9 | 31.5 | 30.0 | 21.4 | 30.7 | 30.7 | 31.0 | 31.4 | 27.1 | 28.0 | 28.2 |
| 0.7 | 0.8 | 12.8 | 15.1 | 14.2 | 12.2 | 14.9 | 12.5 | 14.2 | 12.3 | 11.2 | 12.1 |
| | 0.9 | 21.2 | 25.8 | 19.6 | 22.9 | 22.8 | 22.5 | 23.2 | 21.0 | 20.8 | 21.7 |
| 0.8 | 0.9 | 13.7 | 16.1 | 15.8 | 14.2 | 16.3 | 15.4 | 16.7 | 14.6 | 14.4 | 14.7 |

**Table 5.** Power Simulation under Double Structural Changes: DGP (25).

| $\lambda_1$ | $\lambda_2$ | ME Tests | | | FL | Tests for Double Changes | | | Tests for a Single Change | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $h=\frac{1}{2}$ | $h=\frac{1}{5}$ | $h=\frac{1}{10}$ | Test | MAX-F | AVG-F | EXP-F | MAX-F | AVG-F | EXP-F |
| 0.2 | 0.4 | 13.8 | 15.2 | 12.9 | 13.2 | 14.5 | 13.2 | 14.7 | 12.8 | 11.9 | 12.2 |
| | 0.5 | 17.4 | 17.4 | 15.6 | 14.3 | 16.8 | 15.0 | 16.0 | 13.4 | 13.5 | 13.8 |
| | 0.6 | 19.0 | 19.4 | 15.8 | 15.5 | 18.5 | 16.4 | 18.3 | 14.9 | 14.1 | 14.4 |
| | 0.7 | 23.9 | 21.5 | 16.1 | 17.0 | 22.0 | 19.8 | 21.8 | 17.8 | 15.8 | 17.5 |
| | 0.8 | 24.4 | 20.2 | 16.0 | 16.9 | 21.9 | 20.3 | 22.2 | 17.9 | 16.2 | 17.8 |
| | 0.9 | 23.7 | 18.8 | 15.8 | 14.3 | 20.2 | 20.2 | 21.4 | 17.4 | 13.9 | 16.3 |
| | 1.0 | 22.6 | 18.1 | 15.8 | 14.3 | 19.2 | 19.0 | 21.2 | 17.3 | 13.9 | 16.1 |
| 0.4 | 0.6 | 12.8 | 14.2 | 13.6 | 11.0 | 13.4 | 11.5 | 13.5 | 11.0 | 9.7 | 10.5 |
| | 0.7 | 15.7 | 16.6 | 14.6 | 11.9 | 16.5 | 14.6 | 16.1 | 12.5 | 10.8 | 12.2 |
| | 0.8 | 19.0 | 18.8 | 14.6 | 13.3 | 18.0 | 16.4 | 18.4 | 13.5 | 11.5 | 12.9 |
| | 0.9 | 22.4 | 18.5 | 15.7 | 15.9 | 19.6 | 18.3 | 20.4 | 15.9 | 13.6 | 15.0 |
| | 1.0 | 25.0 | 20.8 | 16.4 | 18.4 | 20.4 | 20.9 | 21.4 | 18.5 | 16.5 | 18.0 |
| 0.6 | 0.8 | 13.8 | 13.6 | 13.2 | 11.7 | 14.4 | 12.6 | 14.3 | 11.7 | 11.3 | 11.8 |
| | 0.9 | 16.7 | 16.6 | 14.4 | 14.8 | 17.3 | 15.0 | 17.0 | 13.7 | 12.6 | 13.8 |
| | 1.0 | 19.3 | 18.8 | 15.9 | 19.0 | 18.3 | 19.2 | 18.5 | 15.8 | 17.1 | 16.9 |
| 0.8 | 1.0 | 13.7 | 14.8 | 13.1 | 15.5 | 14.6 | 15.8 | 16.1 | 16.1 | 15.9 | 16.2 |